The Theory of Musical Temperaments

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## Introduction

From antiquity to modernity, tunings have been immanent to music theory and practice. Beneath the sounds lies a massive body of speculative mathematical structure which has attracted the attention of theorists for thousands of years. Remarkably, the different systems of tuning have also impacted the compositional theories and styles over the ages. In this paper, I will survey several different tuning methods from the ancient Greeks through the late Renaissance. I will then explore the interplay between theoretical temperaments, musical thought, and compositional practice. In this way, I hope to provide a historical perspective on tunings and to point toward a more holistic understanding of harmonic theory. By understanding tunings and their effects on music theory throughout the ages, we can better understand historical theories and develop possible modern analogues.

## Mathematical Preliminaries

The study of music is intimately connected with the study of the physical properties of sound. Thus, a brief explanation of these methods is in order. Sound propagates through air as a mechanical wave. A sound wave has several parameters,[1] but the parameter which determines the human perception of pitch is its frequency—the number of cycles per second. (The unit Hertz, or Hz, is defined as one cycle per second.) Pitches that we perceive as higher have higher frequencies, and vice versa.

[1] E.g., amplitude, frequency, phase.

Musical intervals manifest themselves in the frequency relationship of two sound waves. The human auditory system, perceiving pitch logarithmically, hears intervals not as the difference between the two frequencies, but as the *quotient* between two frequencies. Thus, an interval between a 220 Hz sound and a $440$ Hz sound is perceived the same as the interval between a $440$ Hz sound and an $880$ Hz sound since each of the respective quotients is equal to $2$.

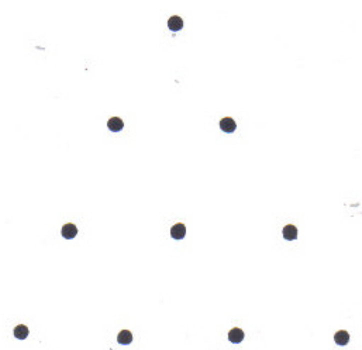
It follows that an increase in pitch by a given interval corresponds to multiplication of the frequency with the fraction corresponding to the frequency ratio of the interval. Conversely, a decrease by a given interval corresponds to the division of the frequency by the corresponding ratio. For example, starting at the pitch $660$ Hz, if we wish to increase this pitch by the interval ratio of $3:2$, we simply compute 660 Hz $\times \frac{3}{2} = 990$ Hz. Or if we wish to decrease this pitch by the same interval, we compute 660 Hz $\div \frac{3}{2} = 660$ Hz $\times \frac{2}{3} = 440$ Hz.

Modern twelve-tone equal temperament takes the interval of an octave, $2:1$, and divides it into twelve equal semitones, each of which we provisionally call $x$. By design, $x$ must satisfy the equation $x^{12} = 2$. Hence, $x = 2^{1/12} \approx 1.059$. We can use these semitones as a yardstick for measuring other intervals. Thus, given an arbitrary frequency ratio $r$, we may enquire as to how many semitones it comprises. This is equivalent to solving the equation $r = (2^{1/12})^{a}$ for $a$: we will then know that $r$ comprises $a$ semitones. We can first simplify and obtain $r = 2^{a/12}$ and then take logarithms, yielding $\log r = \log(2^{a/12})$. Using logarithm properties, we reduce to $\log r = \frac{a}{12} \log 2$. Isolating $a$, we finally obtain $a = 12 \log r / \log 2$. Scientists typically further divide the semitone into $100$ equal units called *cents*, so to convert $a$ from semitones to cents, we multiply the final result by $100$. Thus, in sum, to convert from a frequency ratio $r$ to cents $c$, we may use the formula $c = 1200 \log r / \log 2$.[2]

*Part I: Tuning Systems from Ancient Greece to Renaissance Italy*

## Pythagoras and Numerology

The systematic study of musical temperaments begins with Pythagoras[3] (6th century BCE), who allegedly discovered the connection between musical intervals and numerical ratios by listening to a blacksmith produce consonant and dissonant intervals with different weighted hammers. The Pythagorean brotherhood cultivated a spiritual interest in mathematics. One of their most important symbols was the *tetractys*, which represents the first four positive integers as dots comprising a triangle.



**The Tetractys**

The tetractys possesses many numerological properties. For instance, it represents the four elements: earth, air, fire, and water. The sum of the dots, $1 + 2 + 3 + 4$, is ten, the decimal. But more importantly, the ratios produced by the tetractys correspond to the fundamental

[2] Clarence Barlow, *On Musiquantics* (2011).
[3] Andrew Barker, *Greek Musical Writings*, vol. 2 (Cambridge: Cambridge University Press, 1989), 28–52.

consonances in Pythagorean tuning. That is, by taking strings of the same tension in various proportions, one can create the following consonances: *diapason*, or octave $(2{:}\,1)$, *diapente*, or fifth $(3{:}\,2)$, *diatessaron*, or fourth $(4{:}\,3)$, diapason plus diapente, or twelfth $(3{:}\,1)$, and diapason plus diapason $(4{:}\,1)$.[4] (Note that the interval $4{:}\,2$ is the same as the interval $2{:}\,1$.) By adding and subtracting these fundamental ratios, other intervals can be created. For instance, the difference between the diapente and the diatessaron is $\frac{3}{2} \div \frac{4}{3} = \frac{9}{8}$, the Pythagorean tone. Two tones, or a ditone, is then $\frac{9}{8} \times \frac{9}{8} = \frac{81}{64}$. Then the difference between the diatessaron and the ditone is $\frac{4}{3} \div \frac{81}{64} = \frac{256}{243}$, the Pythagorean limma. A scale can then be constructed with two disjunct tetrachords, as follows:

$$\{C \overset{\text{tone}}{\text{——}} D \overset{\text{tone}}{\text{——}} E \overset{\text{limma}}{\text{——}} F\} \overset{\text{tone}}{\text{——}} \{G \overset{\text{tone}}{\text{——}} A \overset{\text{tone}}{\text{——}} B \overset{\text{limma}}{\text{——}} C\}$$

Thus, to the Pythagoreans, intervals were metaphysically conceived as integer ratios of string lengths on a monochord. Any intervals created outside these parameters (e.g., those that involved prime factors larger than three[5] or those that involved irrational numbers) were rejected. Furthermore, the entire theory of consonance and dissonance relied not on the senses, but on the immanent mathematical structure of intervals. The consonances were all *superparticular* ratios (i.e., ratios of the form $n + 1{:}\,n$) or *multiple* ratios (i.e., ratios of the form $n{:}\,1$). Thus, the diatessaron (fourth), being $4{:}\,3$ and hence superparticular, was a consonance; yet the diapason

[4] These ratios are the reverse of the frequency ratios as mentioned in the mathematical preliminaries—longer string lengths correspond to *lower* notes, and vice versa. But for the purposes of clarity, all ratios in this paper will be written with the first number greater than the second.

[5] For this reason, Pythagorean tuning is sometimes referred to as three-limit tuning.

plus diatessaron (eleventh), being $8:3$ and hence neither superparticular nor multiple, was considered a dissonance.

Unfortunately for the Pythagoreans, there remained internal mathematical conflicts in their tuning system. Arithmetically speaking, the primes $2$ and $3$ have certain incompatibilities. As a consequence of these incompatibilities, the fifths and the octaves will never form a finite chain: if $a$ fifths are to be equal to $b$ octaves, then we necessarily obtain the equation $(\frac{3}{2})^a = 2^b$, or in other words, $3^a = 2^{a+b}$. The latter equation violates the fundamental theorem of arithmetic.[6] As a result, discrepancies must always exist between fifths and octaves if one desires to use the circle of fifths. The *Pythagorean comma* is defined as the discrepancy between twelve fifths and seven octaves: $(\frac{3}{2})^{12} \div 2^7 = \frac{531441}{524288}$. (Another way of obtaining the Pythagorean comma is to start by taking the difference between the tone and the limma: $\frac{9}{8} \div \frac{256}{243} = \frac{2187}{2048}$. This interval is called the *apotome*. Then the Pythagorean comma is the difference between the limma and the apotome: $\frac{2187}{2048} \div \frac{256}{243} = \frac{531441}{524288}$. A third way of finding the comma is to take the difference between six tones and a diapason: $(\frac{9}{8})^6 \div 2 = \frac{531441}{524288}$.) Regardless of how it is computed, the comma represents a tuning problem that theorists struggled to overcome for hundreds of years.

[6] The fundamental theorem of arithmetic asserts that every positive integer greater than $1$ has a unique prime factorization.

**Summary of Pythagorean Intervals**

| INTERVAL | RATIO | CENTS[7] |
|---|---|---|
| Diapason (octave) | $2:1$ | $1200$ |
| Diapente (fifth) | $3:2$ | $702$ |
| Diatessaron (fourth) | $4:3$ | $498$ |
| Pythagorean tone | $9:8$ | $204$ |
| Pythagorean ditone | $81:64$ | $408$ |
| Pythagorean semiditone | $32:27$ | $294$ |
| Limma (minor semitone) | $256:243$ | $90$ |
| Apotome (major semitone) | $2187:2048$ | $114$ |
| Pythagorean comma | $531441:524288$ | $23$ |

## Aristoxenus and Perception

Aristoxenus (4th century BCE) represented a contrasting school of thought to the Pythagoreans. A proponent of acoustical perception over mathematical reason, he attacked his predecessors in his treatise *Elementa Harmonica*, claiming, "[they] speak irrelevantly, ignoring the sense as not being exact, build contrived causes, pretend there are certain ratios of numbers and reciprocal velocities in which the high and the low arise, and propose considerations totally alien to all things and completely opposite to the phenomena."[8] As a result, Aristoxenus judges consonance and dissonance not via immanent mathematical relationships but rather by

[7] To convert from a ratio $r$ to cents $c$, we use the formula derived earlier: $c = 1200 \log r / \log 2$. All calculations of cents are rounded to the nearest integer.

[8] Thomas J. Mathiesen, *Apollo's Lyre: Greek Music and Music Theory in Antiquity and the Middle Ages* (Lincoln: University of Nebraska Press, 2000), 321.

phenomenological principles. This philosophy results in several discrepancies with the Pythagorean system. For instance, Aristoxenus embraces the eleventh as a consonance, proposing on general auditory grounds that any consonant interval added to an octave must yield a consonance. In contrast, the Pythagoreans reject the eleventh not because of auditory concerns, but because its mathematical ratio is neither superparticular nor multiple. Aristoxenus also judges the fourth to be equal to two-and-a-half tones,[9] whereas the Pythagoreans more rigorously define it as two tones and a limma (which is less than half of a tone). Whereas the Pythagoreans prove that the fourth comprises two tones and a limma using the mathematics of ratios, Aristoxenus proves that the fourth is two-and-a-half tones through a complex logical argument which combines both reason and the senses.[10] Mathiesen remarks that although "objections to Aristoxenus's demonstration can be raised on various mathematical grounds, … his demonstration is neither mathematical nor empirical. Rather, it is cast in a totally new spatial logic that mathematical objections cannot address."[11] Aristoxenus's spatial conception of pitch is a radical reworking of the Pythagorean conception: rather than casting pitches as discrete ratios of string lengths, he visualizes pitch as lying along a continuum. This geometric approach is most evident in Aristoxenus's description of the *shades* of the tonoi, where he prescribes pitch bands for the diatonic, chromatic, and enharmonic genera. (For example, a 'normal' diatonic shade would be broken into $1$, $1$, and $1/2$ of a tone measuring from the top down, while a 'low' diatonic shade would be broken into $1\frac{1}{4}$, $3/4$, and $1/2$ of a tone. In principle, many other diatonic shades would be possible, starting with an interval of anywhere between $1$–$1\frac{1}{4}$ of a

[9] Of course, as a corollary to this claim, the octave, comprising a tone and two fourths, must be $1 + 2 \times 2\frac{1}{2} = 6$ tones, or 12 half-tones. Thus, Aristoxenus is arguably on the verge of twelve-tone equal temperament!

[10] Mathiesen 328.

[11] Mathiesen 329.

tone.[12]) The ramifications of a continuum of pitch rather than a discrete set of pitches include the possibility of equal temperament, geometric means, and other constructs which implicitly require irrational numbers (and are thus rejected by the Pythagoreans). Thus, Aristoxenenian harmonics was arguably the forebear of Marchetto's division of the tone, and eventually, mean, whole, and equal temperament.

## Ptolemy and the Middle Path

Ptolemy (2nd century CE) in many ways advocated a synthesis of the Pythagorean and Aristoxenian ideals while also proposing new methods of his own. He viewed the principles of harmonic order as both mathematical (discernible by reason) and phenomenological (regulated by perceptual experience). On the other hand, he criticized aspects of both these schools of thought as well, citing the Pythagoreans' penchant for postulating theoretical relationships not grounded in reality and the Aristoxenians' lack of mathematical precision.

While Ptolemy incorporated some of the Pythagorean tunings, he postulated new ones as well. Criticizing the inconsistency of regarding the fourth as consonant but the eleventh as dissonant, Ptolemy chose to regard both intervals as consonant, arguing that "a note that forms the consonant interval of a fourth with the upper note of an octave forms by extension the same functional interval with the lower note."[13] More dramatically, however, Ptolemy broke from the Pythagorean tetractys, arguing that superparticular ratios beyond $4:3$ should be considered. This was the first historical argument for five-limit tuning. Among the new intervals formed are the *just major third* $(5:4)$, the *just minor third* $(6:5)$, the *minor tone* $(10:9)$, and the *just diatonic semitone* $(16:15)$ As a result, the thirds become more acoustically pure, but due to the immanent

[12] Barker 141–143.
[13] Mathiesen.

mathematical structure additional commas arise between the primes 2, 3, and 5.[14] The comma between 2 and 5 manifests as the difference between one octave and three just major thirds: $2 \div (\frac{5}{4})^3 = \frac{128}{125}$. This interval is called the *lesser diesis*. A comma between 2, 3 and 5 manifests as the difference between the Pythagorean ditone and the just major third: $\frac{81}{64} \div \frac{5}{4} = \frac{81}{80}$. This interval is called the *syntonic comma*. Finally, an additional comma between 2, 3, and 5 manifests as the difference between four just minor thirds and one octave: $(\frac{6}{5})^4 \div 2 = \frac{648}{625}$. This interval is called the *greater diesis*.

**Summary of Ptolemaic Intervals**

| INTERVAL | RATIO | CENTS |
|---|---|---|
| Just major third | $5:4$ | 386 |
| Just minor third | $6:5$ | 316 |
| Minor tone | $10:9$ | 182 |
| Just diatonic semitone | $16:15$ | 112 |
| Lesser diesis | $128:125$ | 41 |
| Syntonic comma | $81:80$ | 22 |
| Greater diesis | $648:625$ | 63 |

Although Ptolemy's tuning allows for greater purity of thirds and fifths, by necessity it introduces more mathematical wrinkles that theorists attempted to work around for hundreds of years. Either as a result of the extra complexity and additional commas or as a result of simple

[14] This fact is again due to the fundamental theorem of arithmetic.

ignorance of Ptolemy's writings, early medieval treatises rejected five-limit tuning in favor of Pythagorean three-limit methods.

## Marchetto, the Divider

As mentioned above, most medieval harmonic treatises accepted the authority of the ancient Greeks and appealed to Pythagorean three-limit tunings, but Marchetto of Padua (14th century) controversially synthesized the Pythagorean tunings with Aristoxenian phenomenology in his treatise, *Lucidarium*. Within, Marchetto proposes the theoretical division of the Pythagorean tone into five equal parts. This division allowed for the creation of two different pairs of semitones: the *diesis* and the *chromatic semitone*, comprising $\frac{1}{5}$ and $\frac{4}{5}$ of the tone, respectively; and the *enharmonic semitone* and the *diatonic* semitone, comprising $\frac{2}{5}$ and $\frac{3}{5}$ of the tone, respectively. The enharmonic semitone and the diatonic semitone were supposed to coincide with the Pythagorean apotome and limma, respectively, while the diesis and the chromatic semitone are new constructs. The audacity of this idea lies in its casual division of the tone into an equal number of parts: the Pythagoreans had long since rejected such ideas on the grounds that any theoretical division of the tone into an equal number of parts requires the use of irrational numbers (i.e., numbers that cannot be written as the quotient of integers). For example, dividing the ratio $9:8$ into five equal parts implies the existence of a ratio $r$ that satisfies the equation $r^5 = \frac{9}{8}$. (The solution $r = 9^{1/5}:8^{1/5}$ was rejected since the numbers involved are irrational.)

**Comparison of Pythagorean Intervals with Marchetto's Intervals**

| INTERVAL | RATIO | CENTS |
|---|---|---|
| Pythagorean limma | $256:243$ | 90 |
| Marchetto's enharmonic semitone | $9^{2/5}:8^{2/5}$ | 82 |
| Pythagorean apotome | $2187:2048$ | 114 |
| Marchetto's diatonic semitone | $9^{3/5}:8^{3/5}$ | 122 |
| Marchetto's diesis | $9^{1/5}:8^{1/5}$ | 41 |
| Lesser diesis | $128:125$ | 41 |
| Marchetto's chromatic semitone | $9^{4/5}:8^{4/5}$ | 163 |
| Greater diesis | $648:625$ | 63 |

In this equal division, Marchetto goes back to Aristoxenus, who had argued for an equal division of the tone into two parts (and, as a corollary, a division of the octave into six tones or twelve parts). Although his idea breaks with Pythagorean tradition, Marchetto is not bold enough to reject its authority. Thus, he attempts to validate his division of the tone via ancient Greek, numerological arguments. The result is flimsy at best (his argument, in essence, proposes that the number nine has particular numerological significance, and since nine is the fifth odd number, thus the tone must be divided into five parts),[15] yet independent of the soundness of the argument, the idea carries manifold practical considerations. It is much easier to imagine dividing a tone into $\frac{2}{5}$ and $\frac{3}{5}$ parts than it is to internalize the Pythagorean ratios of $2187:2048$

[15] Jan Herlinger, "Marchetto's Division of the Whole Tone," *Journal of the American Musicological Society* 34 (1981), 193–216.

and 256: 243. A singer can simply imagine the division as one semitone slightly less than half a tone and the other slightly more. Similarly, for the diesis and chromatic semitones, the singer simply makes a slightly wider discrepancy between the two parts of the tone.

Marchetto's diesis and chromatic semitone are radical new ideas which combine tuning and functional harmony. They are used "so as to color a dissonance such as a third, a sixth, or a tenth striving toward a consonance. It is for the sake of the elegance and beauty of the dissonances that the whole tone is divided beyond the size of the division into the diatonic and enharmonic genera—so that the dissonances may lie closer to the consonances that follow them."[16] For example, if the imperfect major sixth e–c♯ resolved to the perfect octave d–d′, then the semitone between the c♯ and the d′ is taken as the diesis, making the major sixth much wider in this chromatic case than in the diatonic case. In this way, Marchetto allows for a more dynamic conception of intervals, consonance, and dissonance: the same interval (major sixth) will be tuned differently depending on its function (is it diatonic and stable or chromatic and resolving to a perfect consonance?). Thus, the concept of consonance and dissonance breaks down into two categories: auditory and functional. Although Marchetto does not take this idea to its logical conclusion, he sows the first seeds of functional thought.

## Zarlino, Thirds, and the Mean-Tone

As music became progressively more chromatic, exploring more and more key areas, the various commas between octaves, thirds, and fifths became more prominent. As a result, theorists began to explore possible alternate temperaments which would achieve a greater flexibility throughout several key areas. Unfortunately, no single solution stood out as best: the

[16] Marchetto da Padova, *Lucidarium*, trans. Jan Herlinger (Chicago: University of Chicago Press, 1985).

better in tune the thirds are, the worse the fifths, and vice versa. *Mean-tone* temperament was one popular possibility starting in the 16th century. The basic idea was to start with the just major third and take its geometric mean with the prime to create a tone which is exactly half of this third. Since the thirds are now pure, the fifths must be tempered in order to account for the syntonic comma. One solution, discussed by Pietro Aron[17] (1523) and Gioseffo Zarlino (1558),[18] is to temper each of the fifths flat by $1/4$ comma. In other words, we take the $3:2$ fifth and lower it by $1/4$ of the $81:80$ comma: $\frac{3}{2} \div \sqrt[4]{\frac{81}{80}} = \sqrt[4]{5}$. The fourths are correspondingly tempered sharp by $1/4$ comma, being taken as the difference between the octave and the mean-tone fifth: $2 \div \sqrt[4]{5} = \frac{2}{\sqrt[4]{5}}$. The minor third is taken as the difference between the mean-tone fifth and the just major third: $\sqrt[4]{5} \div \frac{5}{4} = \frac{4}{5^{3/4}}$. (Since the mean-tone fifth is flat from the acoustically pure fifth by $1/4$ comma, it follows that the mean-tone minor third must also be $1/4$ comma flat from the acoustically pure minor third.) The diatonic semitone is the difference between the mean-tone fourth and the just major third: $\frac{2}{\sqrt[4]{5}} \div \frac{5}{4} = \frac{8}{5^{5/4}}$. Finally, the chromatic semitone is the difference between the mean-tone tone and the diatonic semitone: $\frac{\sqrt{5}}{2} \div \frac{8}{5^{5/4}} = \frac{5^{7/4}}{16}$.

[17] Pietro Aron, *Thoscanello de la musica*, trans. P. Bergquist (Colorado Springs: Colorado College Music Press, 1970).

[18] Gioseffo Zarlino, *Istitutioni harmoniche* (1558).

**Summary of 1/4 Mean-Tone**

| INTERVAL | RATIO | CENTS |
|---|---|---|
| Just major third | $5:4$ | 386 |
| 1/4 mean-tone tone | $\sqrt{5}:2$ | 193 |
| 1/4 mean-tone fifth | $\sqrt[4]{5}$ | 697 |
| 1/4 mean-tone fourth | $2:\sqrt[4]{5}$ | 503 |
| 1/4 mean-tone minor third | $4:5^{3/4}$ | 310 |
| 1/4 mean-tone diatonic semitone | $8:5^{5/4}$ | 117 |
| 1/4 mean-tone diatonic semitone | $5^{7/4}:16$ | 76 |

However, this system does not address the Pythagorean comma between fifths and octaves. Twelve mean-tone fifths are approximately 41 cents short of seven octaves, so in order to link the fifths and the octaves, one of the fifths must be 41 cents sharper than the other. This fifth is called the 'wolf fifth' since it is so out of tune it howls like a wolf. The common practice of the time was to have the G♯ be 41 cents below the A♭ in a tuning cycle starting at C, since those pitches belonged to more remote key areas. Still, the compromises involved in this tuning system were not universally accepted. Other mean tone tuning systems were also proposed, including the 2/7 comma system as advocated also by Zarlino.[19] In this system, Zarlino rectifies one of the problems of 1/4 mean-tone: the major thirds are pure, but the minor thirds are flat. In the 2/7 system, each of the thirds is tempered 1/7 comma flat, as a compromise. Thus the fifth is tempered flat by 2/7 comma, slightly more than the 1/4 comma temperament. The other intervals are derived analogously as before. Thus, neither the fifths nor the thirds are pure, but

[19] Zarlino.

rather, all the intervals are slightly compromised in order to facilitate greater ease of modulation. This shift from pure intervals but only in one key area to somewhat impure intervals but in all key areas marked the beginning of the idea of equal temperament, which was soon advocated by Vincenzo Galilei and soon proliferated on fretted instruments in the Baroque era.

**Summary of $2/7$ Mean-Tone**

| INTERVAL | RATIO | CENTS |
|---|---|---|
| $2/7$ mean-tone major third | $5^{8/7}: 2^{10/7}3^{4/7}$ | 383 |
| $2/7$ mean-tone tone | $5^{4/7}: 2^{5/7}3^{2/7}$ | 192 |
| $2/7$ mean-tone fifth | $2^{1/7}5^{2/7}: 3^{1/7}$ | 696 |
| $2/7$ mean-tone fourth | $2^{6/7}3^{1/7}: 5^{2/7}$ | 504 |
| $2/7$ mean-tone minor third | $2^{11/7}3^{3/7}: 5^{6/7}$ | 313 |
| $2/7$ mean-tone diatonic semitone | $2^{16/7}3^{5/7}: 5^{10/7}$ | 121 |
| $2/7$ mean-tone chromatic semitone | $25: 24$ | 71 |

*Part II: The Effect of Tuning Paradigms on Harmonic Thought*

## The Cult of Pythagoras

It is difficult to overstate the stranglehold of Pythagorean mathematics and tuning on early medieval musical thought. Scholars of the time relied on the ancient Greeks as high authorities, and hardly any were prepared to challenge the dogma concerning the mathematical aspects of pitch and interval. Relying on the paradigm of the tetractys and the emphasis on superparticular ratios and arithmetic/harmonic means, virtually all the medieval treatises advocated three-limit tuning and codified the consonances as the unison, octave, fifth, and fourth, in that order. Indeed, most musical treatises contained large sections on arithmetic (number theory) and numerology. The list includes Boethius's *De Institutione Musica* (6th century), which had enormous influence on future theoretical writings; the anonymous *Scolica Enchiriadis* (9th century); and the anonymous *Alia Musica* (10th century).

Thirds (i.e., Pythagorean thirds) were universally considered dissonances. According to Jeremy Montagu, since the Pythagorean third is so much sharper than the acoustically pure third (a difference of approximately twenty-two cents, the syntonic comma), "the third was regarded as a dissonance in the Middle Ages simply because it was indeed dissonant."[20] Whether it was due to mathematical properties or perceptual properties, the status of the third did not change until the early Renaissance. By this time other paradigms were slowly arising. But Pythagoras did not die out easily—even as late as 1492, Franchino Gaffurio's treatise *Theorica musicae* continued to espouse Pythagorean doctrine, as taken from Boethius.

### Aristoxenus and Marchetto Strike Back

Although Aristoxenus was largely unknown to medieval theorists, his emphasis on perception and pragmatism was coincidentally upheld by some of the more radical thinkers. In particular, Marchetto (14th century) began the shift away from speculative theory in favor of more practical theory. The philosophy of regarding the tone not as a theoretical ratio of integers but rather as a band capable of division into parts reflects this paradigm shift and empowers singers to more easily grasp the distinction between the various types of semitone. More

[20] Jeremy Montagu, "Temperament," In *The Oxford Companion to Music*, ed. Alison Latham, *Oxford Music Online*, http://www.oxfordmusiconline.com/subscriber/article/opr/t114/e6695.

importantly, the implications of this new model for future tunings such as mean-tone and equal temper can hardly be overstated. Musical tunings were finally evolving beyond the three-limit Pythagorean model. Hand in hand with this evolution, the style of composition was also slowly morphing.

The gradual change in style from monody to polyphony encouraged the use of more and more harmonic intervals, which naturally included sixths and thirds. The 14th century *ars nova* saw increasing emphasis away from melody to harmony. Marchetto's distinction between diatonic harmonies and functional harmonies added considerable nuance to the theory of consonance and dissonance. Soon thereafter, the status of the fourth became controversial. For instance, Tinctoris classifies the fourth as a dissonance, breaking further with the Pythagorean thought.[21] The reason for this new attitude toward the fourth has little to do with the quality of the fourth in a vacuum. Rather, as the idea of harmonic function became more prevalent, and the use of thirds and sixths as stable sonorities proliferated, the fourth became less stable and more frequently resolved to a third. Naturally, as the thirds became more prominent, new tuning systems were devised to allow for more pure thirds, and the five-limit theories of Ptolemy became more widespread.

## Zarlino Looks Forward

Zarlino's extensive tuning methods influenced many ideas in harmonic theory and helped to bring about the model of equal temper. He explains three different tunings: the Ptolemaic five-limit tuning, the $1/4$ mean-tone system, and the $2/7$ mean-tone system. Each of these tunings emphasizes the importance of thirds more so than any previous method; consequently, thirds in

[21]Johannes Tinctoris, *Liber de arte contrapuncti* (1477), trans., ed. Albert Seay (Rome: American Institute of Musicology, 1961).

compositional practice gained much importance. Many canonical cadences ended with what modern theorists would call a major or minor triad, and Zarlino mentions these (though not by name) as well in his treatises. The mean-tone systems naturally promoted the possibility of music which modulated through remote key areas. The eventual development of well and equal temperament in the Baroque era owes its impetus to Zarlino's innovations.

In addition, through his careful mathematical development of tunings, Zarlino came close to discovering the idea of interval *inversion*: the dual relationship between fourths/fifths, minor thirds/major sixths, etc. Yet by considering the major sixth to be a composite consonance (i.e., the sum of a major third and a fourth) instead of as a complementary consonance (i.e., the difference between an octave and a minor third), he blinded himself to further theoretical possibilities, as Robert Wienpahl notes.[22] Being so close to other important theoretical constructs of functional tonality such as the conception of the triad, Zarlino paved the way for a bountiful new line of speculative musical thought.

## Postscript

The modern age of technology has seen a burst of activity on the front of temperament. "Microtonality" has been explored in a variety of fashions—from different temperaments (19-, 31-, 41-, and 53-equal temper have all been proposed) to more advanced just intonation (seven-limit and beyond). Our department's own Professor Clarence Barlow has devised a mathematical theory of 'harmonicity,' which attempts to rigorously define sensory and harmonic consonance and dissonance.[23] The increased scientific understanding of the overtone series led to an entire

[22] Robert W. Wienpahl, "Zarlino, the Senario, and Tonality," *Journal of the American Musicological Society* 12.1 (Spring, 1959), 31.
[23] From Professor Barlow's course MUS 211B: "Pitch as a Compositional Tool."

'spectralist' movement in the 1970s.[24] The interplay between tunings, theory, and practice has continued throughout the ages and most likely will not end anytime soon.

**Summary of Intervals Discussed, in Order of Size**

| INTERVAL | RATIO | CENTS |
|---|---|---|
| Syntonic comma | $81:80$ | 22 |
| Pythagorean comma | $531441:524288$ | 23 |
| Marchetto's diesis | $9^{1/5}:8^{1/5}$ | 41 |
| Lesser diesis | $128:125$ | 41 |
| Greater diesis | $648:625$ | 63 |
| $2/7$ mean-tone chromatic semitone | $25:24$ | 71 |
| $1/4$ mean-tone diatonic semitone | $5^{7/4}:16$ | 76 |
| Marchetto's enharmonic semitone | $9^{2/5}:8^{2/5}$ | 82 |
| Pythagorean limma | $256:243$ | 90 |
| Equal tempered semitone | $2^{1/12}:1$ | 100 |
| Just diatonic semitone | $16:15$ | 112 |
| Pythagorean apotome | $2187:2048$ | 114 |
| $1/4$ mean-tone diatonic semitone | $8:5^{5/4}$ | 117 |
| $2/7$ mean-tone diatonic semitone | $2^{16/7}3^{5/7}:5^{10/7}$ | 121 |
| Marchetto's diatonic semitone | $9^{3/5}:8^{3/5}$ | 122 |
| Marchetto's chromatic semitone | $9^{4/5}:8^{4/5}$ | 163 |

[24] Julian Anderson, "Spectral music." In *Grove Music Online*. *Oxford Music Online*, http://www.oxfordmusiconline.com/subscriber/article/grove/music/50982.

| | | |
|---|---|---|
| Minor tone | $10:9$ | 182 |
| $2/7$ mean-tone tone | $5^{4/7}:2^{5/7}3^{2/7}$ | 192 |
| $1/4$ mean-tone tone | $\sqrt{5}:2$ | 193 |
| Equal tempered tone | $2^{2/12}$ | 200 |
| Pythagorean tone | $9:8$ | 204 |
| Pythagorean semiditone | $32:27$ | 294 |
| Equal tempered minor third | $2^{3/12}$ | 300 |
| $1/4$ mean-tone minor third | $4:5^{3/4}$ | 310 |
| $2/7$ mean-tone minor third | $2^{11/7}3^{3/7}:5^{6/7}$ | 313 |
| Just minor third | $6:5$ | 316 |
| $2/7$ mean-tone major third | $5^{8/7}:2^{10/7}3^{4/7}$ | 383 |
| Just major third | $5:4$ | 386 |
| Equal tempered major third | $2^{4/12}$ | 400 |
| Pythagorean ditone | $81:64$ | 408 |
| Diatessaron (fourth) | $4:3$ | 498 |
| Equal tempered fourth | $2^{5/12}$ | 500 |
| $1/4$ mean-tone fourth | $2:\sqrt[4]{5}$ | 503 |
| $2/7$ mean-tone fourth | $2^{6/7}3^{1/7}:5^{2/7}$ | 504 |
| $2/7$ mean-tone fifth | $2^{1/7}5^{2/7}:3^{1/7}$ | 696 |
| $1/4$ mean-tone fifth | $\sqrt[4]{5}$ | 697 |
| Equal tempered fifth | $2^{7/12}$ | 700 |

| | | |
|---|---|---|
| Diapente (fifth) | $3:2$ | 702 |
| Diapason (octave) | $2:1$ | 1200 |

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